Linearity and Transformations

EC 320: Introduction to Econometrics

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Goals for Today

- Clarify what “linear” means in regression
- Distinguish **linear in variables** vs **linear in parameters**
- Learn how to redefine variables to restore linearity
- Apply log and exponential transformations
- Interpret **elasticities** in transformed models

Motivation

- OLS assumes linearity in parameters
- Linear models are easier to estimate and interpret
- Example: a 1-unit change in X causes a β-unit change in Y

The Usual Model

$$Y_i = \beta_0 + \beta_1 X_{1i} + \cdots + \beta_k X_{ki} + u_i$$

Two meanings of "linear":

1. Linear in variables
2. Linear in parameters

Linear in Variables

- Variables appear untransformed
- Example: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2$
- If variables enter as X^2 or $\sqrt{X}$ $\rightarrow$ not linear in variables
- Fix by redefining variables

Example: Nonlinear Variables

$$Y = \beta_0 + \beta_1 X_1^2 + \beta_2 \sqrt{X_2} + \beta_3 \ln X_3$$

Define new variables:

$$Z_1 = X_1^2, \quad Z_2 = \sqrt{X_2}, \quad Z_3 = \ln X_3$$

Then:

$$Y = \beta_0 + \beta_1 Z_1 + \beta_2 Z_2 + \beta_3 Z_3$$

Example: Diminishing Returns to Humor

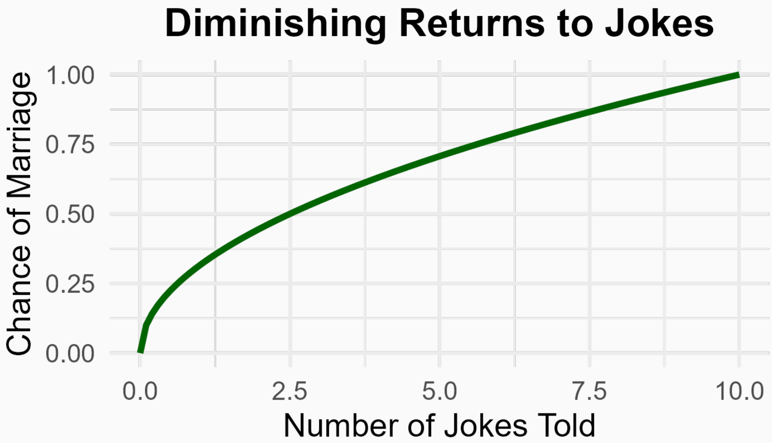
- Suppose you estimate the impact of **joke-telling** on the probability of marriage.
  - At first, more jokes may help your chances...
  - ...but at some point, too many jokes might backfire.
- We can model this by including both *Jokes* and *Jokes*^{2}:

$$Marriage_i = \alpha_0 + \alpha_1 Jokes_i + \alpha_2 Jokes_i^2 + u_i$$

- Here, α_1 captures the initial effect of humor, and α_2 allows the effect to taper off or reverse.

Graphically

The relationship might look something like this:



Key Takeaway

Rule
If a variable enters nonlinearly, redefine it. For any $f(X)$, set $Z = f(X)$ to preserve linearity.

Nonlinearity in Parameters

- More problematic: parameters appear inside exponents or logs
- Example: Cobb–Douglas
$$Y = \beta_0 X_1^{\beta_1} X_2^{\beta_2}$$
- β_1, β_2 enter nonlinearly $\rightarrow$ not suitable for OLS as-is

Group Activity #1

Compare:

$$(1)\ Y = \beta_0 + \beta_1 X + \varepsilon$$
$$(2)\ Y = \beta_0 + \beta_1 X^{\beta_2} + \varepsilon$$

- Take $\frac{dY}{dX}$ for each
- Discuss: how does linearity in parameters differ?

Logarithmic Transformation

Start from:

$$Y = \beta_0 X^{\beta_1}$$

Differentiate:

$$\frac{dY}{dX} = \beta_0 \beta_1 X^{\beta_1 - 1}$$

Substitute $Y = \beta_0 X^{\beta_1}$:

$$\frac{dY}{dX} = \beta_1 \frac{Y}{X}$$

Elasticity Connection

$$\text{Elasticity} = \frac{dY/dX}{Y/X} = \beta_1$$

- β_1 measures the % change in Y for a 1% change in X
- Example: if $\beta_1 = 5$, then a 1% increase in $X \rightarrow$ 5% increase in Y

Linearizing with Logs

Take logs:

$$\ln Y = \ln \beta_0 + \beta_1 \ln X$$

Define:

$$Y^* = \ln Y, \quad X^* = \ln X, \quad \beta_0^* = \ln \beta_0$$

Then:

$$Y^* = \beta_0^* + \beta_1 X^*$$

Why Logs Matter

- Converts nonlinear model into linear form
- Coefficients become elasticities
- Often stabilizes variance and linearizes curvature

Group Activity #2: Semi-Log Relationship

- Consider a model where the outcome grows **exponentially** with X:

$$Y = \beta_1 e^{\beta_2 X}$$

- Work with your group to:
    1. Take $\frac{dY}{dX}$
    2. Interpret β_2 in terms of the rate of change of Y
    3. Apply a log transformation to make it linear

Semi-Log Model: Interpretation

$$
\begin{aligned}
\frac{dY}{dX} &= \beta_1 \beta_2 e^{\beta_2 X} = Y \beta_2 \\
\Rightarrow \frac{1}{Y}\frac{dY}{dX} &= \beta_2
\end{aligned}
$$

- β_2 is the **proportional (percent) change in** Y for a one-unit increase in X.
- This is called a **semi-log model** because only Y is logged.

Semi-Log Interpretation

Taking logs of both sides:

$$\ln Y = \ln \beta_1 + \beta_2 X$$

Define:

$$Y^* = \ln Y, \quad \beta_1^* = \ln \beta_1$$

$$\Rightarrow Y^* = \beta_1^* + \beta_2 X$$

Semi-Log Model: Economic Meaning

- A one-unit increase in X changes Y by approximately $100 \times \beta_2$ percent.
- Example: if $\beta_2 = 0.05$, a one-unit increase in X increases Y by 5%.
- Commonly used for:
  - Wage–education models (ln Wage vs. Years of Education)
  - Price and income growth models

Summary

- Linearity can mean variables or parameters
- Redefine variables to fix nonlinear inputs
- Use logs to linearize power or exponential models
- Interpret coefficients in log–log models as elasticities

Reflection Questions (Groups)

1. How would you estimate $Y = \beta_0 + \beta_1 X^2$ linearly?
2. Why does $Y = \beta_0 X^{\beta_1}$ violate linearity in parameters?
3. In a log–log model, what does β_1 measure?
4. What is β_2's meaning in $Y = \beta_1 e^{\beta_2 X}$?
5. When might you prefer a semi-log over a log–log model?